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*In cherished memory of A.P. Favorskii*

# Particle Method for Electrons in a Scattering Medium

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**Abstract**—The Cauchy problem for the kinetic and electrodynamic equations describing the propagation of an electron flow in a scattering medium and generation of self-consistent electromagnetic field is considered. The electron distribution function is defined in the space of finitely supported generalized functions. Algorithms for the simulation of scattering in the approximation of single and multiple collisions in a time step are presented. Specificities of application of this algorithm in a dense scattering medium and ionized region of large volume are considered.

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## INTRODUCTION

The study of propagation of ionizing radiation in gaseous and condensed media is important for many areas of science and engineering, e.g., research in the properties of matter in extreme conditions [1], cosmic radiation [2], etc. Mathematical modeling is an efficient tool for such studies, because it enables one to optimize expensive experimental works.

In the general case, scattering of photons and electrons with a high energy substantially exceeding the ionization potential of the medium leads to formation of a cascade of particles. This results from collisions of particles with atoms of the medium and the ensuing degradation of the spectrum. Photons suffer Compton scattering and photoabsorption and form electron–positron pairs [3–6]. Fast electrons and positrons produce bremsstrahlung, impact ionization, and molecular excitation and suffer elastic scattering [5–8]. In addition, positrons annihilate on atomic electrons with ensuing photon emission [4, 5]. The formation of charged particles—electrons, positrons, and ions—is accompanied by charge separation and produces electromagnetic field [8]. At a sufficient flux density, the plasma and Larmor electron and positron frequencies become comparable with the collision frequency. The electromagnetic field becomes self-consistent and affects the charged particle motion, which leads to instabilities of their flow. As a result of impact ionization, in the continuous spectrum, in addition to fast charged particles, low-energy electrons are formed [9, 10]. Under the action of self-consistent field, these electrons acquire a directed velocity and form an additional electric current.

The physical phenomenon considered above is described by quasilinear kinetic equations for the distribution function of cascade particles and Maxwell's equations for the self-consistent electromagnetic field. Two basic approaches to the numerical solution of mathematical problems for the kinetic equations are being developed. The first one, in fact, is a finite-difference approach and is termed the *discrete-ordinates method* [11–13]. The second approach, implemented in the Monte Carlo [14, 15] and particle [16, 17] methods, consists in the simulation of a dynamic system by constructing a generalized solution of the kinetic equation. Both approaches have their advantages and drawbacks. The finite-difference methods are more accurate and make possible the direct calculation of the distributions in the space of differentiable functions. The dynamic methods converge much more slowly and require the computation of functionals over the space of generalized solutions. On the other hand, the dynamic methods make possible an efficient program implementation on parallel supercomputers.

The authors of the present work succeeded in obtaining results of certain practical significance on the basis of the second approach [18, 19]. The experience of its application has demonstrated that one of important problems is to combine the simulation of collisions and transport of charge particles in a self-consistent electromagnetic field. In this connection, the current work offers a numerical algorithm for

solving the kinetic equation for the simulation of collisions. The method of numerical solution of Maxwell's equations was thoroughly considered in [20], and the algorithm implementing the particle methods in the absence of collisions or in the mean energy loss approximation, in [21–23].

The complexity of simulation of electron flows substantially exceeds the complexity of simulation of photon transport. The reason is that electrons are under the action of the self-consistent electromagnetic field and scattering is determined by the long-range Coulomb interaction with the medium. Therefore, in this work, the kinetic equation for electrons is considered. The algorithms for simulation of electron collisions are extended to the case of positrons and photons; they can also be applied for describing the cascade as a whole.

## 1. PROBLEM STATEMENT

Consider an electron distribution function $f = f(t, \mathbf{r}, \mathbf{p})$ in the phase space $(\mathbf{r}, \mathbf{p}) = \mathbb{R}^3_{\mathbf{r}} \times \mathbb{R}^3_{\mathbf{p}}$ of coordinates $\mathbf{r} = (x, y, z)$ and momenta $\mathbf{p} = (p_x, p_y, p_z)$. This functions obeys the equations

$$\frac{\partial f}{\partial t} + \operatorname{div}_{\mathbf{r}}(\mathbf{v}f) + e\operatorname{div}_{\mathbf{p}}[(\mathbf{E} + [\boldsymbol{\beta}, \mathbf{H}])f] + \sigma^t \nu f = Q(t, \mathbf{r}, \mathbf{p}) + \int d\mathbf{p}' \sigma(\mathbf{p}, \mathbf{p}') \nu' f(\mathbf{p}'), \tag{1}$$

$$\operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \quad \operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \tag{2}$$

where $t$ is the laboratory time; $\mathbf{v}$ is the electron velocity; $\mathbf{E} = \mathbf{E}(t, \mathbf{r})$ and $\mathbf{H} = \mathbf{H}(t, \mathbf{r})$ are the electric and magnetic fields, respectively; $\operatorname{div}_{\mathbf{r}}$ and $\operatorname{div}_{\mathbf{p}}$ denote divergence in the coordinate and momentum spaces, respectively; $e$ is the electron charge; $\boldsymbol{\beta} = \mathbf{v}/c$, $c$ s the speed of light in vacuum; $\sigma^t$ is the total macroscopic electron scattering cross section; $\sigma(\mathbf{p}, \mathbf{p}')$ is the differential macroscopic electron scattering cross section; $\mathbf{p}'$ and $\mathbf{p}$ are the electron momenta before and after a collision, respectively; and $\mathbf{j}$ is the electron current density. The external electron source is defined by the function $Q = Q(t, \mathbf{r}, \mathbf{p})$ on the right of Eq. (1).

Following the particle method (see [16]), the solution to Eq. (1) is sought in the class of finitely supported generalized functions (see [24]) defined in the test function space of finitely supported infinitely differentiable functions $\varphi = \varphi(\mathbf{r}, \mathbf{p})$ in the phase space $(\mathbf{r}, \mathbf{p})$ and depending on the parameter $t \geq 0$.

The requirement for the function $f(t, \mathbf{r}, \mathbf{p})$ to be finitely supported is necessary for the correct definition of the electric current density $\mathbf{j} = \mathbf{j}(t, \mathbf{r})$ in Maxwell's equations (2). Let us define it as follows. The function $\mathbf{v}f(t, \mathbf{r}, \mathbf{p})$is finitely supported since the function $f(t, \mathbf{r}, \mathbf{p})$ is finitely supported and $\mathbf{v}$ is continuously differentiable (see [24]). Let us define the current density as the action of the finitely supported generalized function $\mathbf{v}f(t, \mathbf{r}, \mathbf{p})$ on a infinitely differentiable function $W(|\mathbf{r} - \boldsymbol{\alpha}|, \Delta)$, $\boldsymbol{\alpha} \in \mathbb{R}^3_{\mathbf{r}}$, $\Delta > 0$, satisfying the conditions

$$\int_{\mathbb{R}^3_{\mathbf{r}}} W(|\mathbf{r} - \boldsymbol{\alpha}|, \Delta) d\boldsymbol{\alpha} = 1, \quad \lim_{\Delta \to 0} \int_{\mathbb{R}^3_{\mathbf{r}}} d\mathbf{r} W(|\mathbf{r} - \boldsymbol{\alpha}|, \Delta) \varphi(\mathbf{r}, \mathbf{p}) = \varphi(\boldsymbol{\alpha}, \mathbf{p}).$$

Despite that the function $W(|\mathbf{r} - \boldsymbol{\alpha}|, \Delta)$ is not finitely supported in the space $(\mathbf{r}, \mathbf{p})$, the current density and the electron density $n = n(t, \mathbf{r})$ are defined correctly just because the function $f(t, \mathbf{r}, \mathbf{p})$ is finitely supported:

$$\mathbf{j} = (f(t, \boldsymbol{\alpha}, \mathbf{p}), \mathbf{v}W(|\mathbf{r} - \boldsymbol{\alpha}|, \Delta)), \quad n = (f(t, \boldsymbol{\alpha}, \mathbf{p}), W(|\mathbf{r} - \boldsymbol{\alpha}|, \Delta)). \tag{3}$$

In such definition of the current density, the electric and magnetic fields calculated from Maxwell's equations (2) are infinitely differentiable functions of the coordinates. Therefore, the term $\operatorname{div}_{\mathbf{r}}(\mathbf{v}f) + e\operatorname{div}_{\mathbf{p}}[(\mathbf{E} + [\boldsymbol{\beta}, \mathbf{H}])f]$ on the left of Eq. (1) is a finitely supported generalized function (see [24]).

Represent the collision integral in Eq. (1) in the form

$$\sigma^t \nu f - \int d\mathbf{p}' \sigma(\mathbf{p}, \mathbf{p}') \nu' f(\mathbf{p}') = \int d\mathbf{r}' \int d\mathbf{p}' \delta(\mathbf{r}' - \mathbf{r})[\sigma(\mathbf{p}', \mathbf{p}) \nu f(\mathbf{p}) - \sigma(\mathbf{p}, \mathbf{p}') \nu' f(\mathbf{p}')]$$

and consider the function $\sigma(\mathbf{p}, \mathbf{p}')\delta(\mathbf{r} - \mathbf{r}')$. The differential cross section $\sigma(\mathbf{p}, \mathbf{p}')$ for any of the considered scattering processes depends on the cosine of the angle between the vectors $\mathbf{p}'$ and $\mathbf{p}$ and on the absolute values of the momenta before and after scattering. The cross sections vanish for $p > p'$, because the energy of a particle cannot increase as a result of scattering. Since scattering without changing the momentum is

possible (for example, the ion charge exchange), the differential cross section can take the form $\sigma(\mathbf{p},\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}')$. For these reasons, the aggregate $\sigma(\mathbf{p},\mathbf{p}')\delta(\mathbf{r}-\mathbf{r}')$ may be considered as a finitely supported generalized function of variables $\mathbf{p}'$ and $\mathbf{r}'$ with the first order of singularity. In this case, the collision integral is a convolution of finitely supported generalized functions, i.e., a finitely supported generalized function. The orders of singularity of all terms of the equations as functions of the variables $\mathbf{r}$ and $\mathbf{p}$ coincide.

Thus, the external electron source $Q = Q(t,\mathbf{r},\mathbf{p})$ on the right of Eq. (1) must also be defined by a finitely supported generalized function of the variables $\mathbf{r}$ and $\mathbf{p}$.

The differential operators of Eqs. (1) and (2) define a quasilinear system of first-order hyperbolic partial differential equations. For this system, we consider the Cauchy problem (see [25]) with homogeneous initial data at zero time.

The representation of the solution of the kinetic equation by a finitely supported generalized function was considered in [21, 23] and will be improved in the present work.

Equation (1) is equivalent to the integral equation

$$f = \int_0^t d\tilde{t}\int d\tilde{\mathbf{r}}\int d\tilde{\mathbf{p}}\Big[Q(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}) + \int d\mathbf{p}'\sigma(\tilde{\mathbf{p}},\mathbf{p}')v'f(\tilde{t},\tilde{\mathbf{r}},\mathbf{p}')\Big]\exp\left\{-\int_{\tilde{t}}^t dt'\sigma^t v^{s'}\right\}\delta\left(\mathbf{r}-\mathbf{r}^s\right)\delta\left(\mathbf{p}-\mathbf{p}^s\right), \tag{4}$$

where the functions $\mathbf{r}^s = \mathbf{r}^s(t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})$ and $\mathbf{p}^s = \mathbf{p}^s(t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})$ are solutions of the equations of motion

$$\frac{d\mathbf{r}^s}{dt} = \mathbf{v}^s, \quad \frac{d\mathbf{p}^s}{dt} = e\left(\mathbf{E}\left(t,\mathbf{r}^s\right) + \frac{1}{c}\left[\mathbf{v}^s,\mathbf{H}\left(t,\mathbf{r}^s\right)\right]\right) \tag{5}$$

with the initial conditions $\mathbf{r}^s\big|_{t=\tilde{t}} = \tilde{\mathbf{r}}$ and $\mathbf{p}^s\big|_{t=\tilde{t}} = \tilde{\mathbf{p}}$. The integrand in the exponent has the form $\sigma^t v^{s'} \equiv \sigma^t\left(p^{s'}\right)v\left(p^{s'}\right)$, where $\mathbf{p}^{s'} = \mathbf{p}^s(t',\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})$.

The equivalence of Eqs. (1) and (4) is justified by substituting (4) into (1) and considering the action of the residual term on the test function from the test function space. This procedure was presented in [23] for collisionless equation (1) and is not repeated here. It should only be added that the derivative on the right of (4) with respect to the upper limit of integration with respect to the variable $\tilde{t}$ is $Q(t,\mathbf{r},\mathbf{p}) + \int d\mathbf{p}'\sigma(\mathbf{p},\mathbf{p}')v'f(t,\mathbf{r},\mathbf{p}')$.

## 2. APPROXIMATION OF SINGLE SCATTERING IN A TIME STEP

Suppose that functions $f \equiv f(t,\mathbf{r},\mathbf{p})$, $\mathbf{E} = \mathbf{E}(t,\mathbf{r})$, and $\mathbf{H} = \mathbf{H}(t,\mathbf{r})$ are the solution to the Cauchy problem (1) and (2) with zero initial conditions. Consider $f \equiv f(t+\Delta t,\mathbf{r},\mathbf{p})$ in accordance with (4):

$$\begin{aligned} f(t+\Delta t) = &\int_t^{t+\Delta t} d\tilde{t}\int d\tilde{\mathbf{r}}\int d\tilde{\mathbf{p}}\exp\left\{-\int_{\tilde{t}}^{t+\Delta t} dt'\sigma^t v^{s'}\right\}\Phi(\mathbf{r},\mathbf{p},t+\Delta t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}) \\ &\times\Big[f(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})\delta(t-\tilde{t}) + Q(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}) + \int d\mathbf{p}'\sigma(\tilde{\mathbf{p}},\mathbf{p}')v'f(\tilde{t},\tilde{\mathbf{r}},\mathbf{p}')\Big], \end{aligned} \tag{6}$$

where $\Phi(\mathbf{r},\mathbf{p},t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}) \equiv \delta\left(\mathbf{r}-\mathbf{r}^s(t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})\right)\delta\left(\mathbf{p}-\mathbf{p}^s(t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})\right)$.

Formula (6) describes electron propagation and absorption in the time interval $(t,t+\Delta t)$. It sums electrons formed before the instant of time $t$ (the first term in brackets on the right-hand side), electrons produced by the source (the second terms), and electrons forming as a result of collisions (the third term) in the time interval $(t,t+\Delta t)$.

The approximation of single scattering of particles in a time step $\Delta t$ consists in the replacement of the right-hand side of Eq. (6) by the following aggregate:

$$\tilde{f}(t+\Delta t) = \int_t^{t+\Delta t} d\tilde{t} \int d\tilde{\mathbf{r}} \int d\tilde{\mathbf{p}} \exp\left\{-\int_{\tilde{t}}^{t+\Delta t} dt' \sigma^t v^{s'}\right\} \Phi(\mathbf{r},\mathbf{p},t+\Delta t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})$$
$$\times \left[ f(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})\delta(t-\tilde{t}) + Q(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}) + \delta(t-\tilde{t}) \int_{t-\Delta t}^{t} dt' \int d\mathbf{p}' \sigma(\tilde{\mathbf{p}},\mathbf{p}') v' f(t',\tilde{\mathbf{r}},\mathbf{p}') \right]. \quad (7)$$

The first and second terms on the right of (7) coincide with the corresponding terms in (6). The third term describes the propagation and absorption in the time interval $(t,t+\Delta t)$ of particles formed in the time interval $(t-\Delta t,t)$ due to scattering. In this case, we assume that all acts of scattering took place at the instant of time $t$.

To complete the definition of the approximation process, it should be noted that

$$f(\Delta t) = \int_0^{\Delta t} d\tilde{t} \int d\tilde{\mathbf{r}} \int d\tilde{\mathbf{p}} Q(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}) \exp\left\{-\int_{\tilde{t}}^{\Delta t} dt' \sigma^t \left(\mathbf{r}^{s'}, p^{s'}\right) v^{s'}\right\} \Phi(\mathbf{r},\mathbf{p},\Delta t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}). \quad (8)$$

Consider the difference of (7) and (6):

$$\Delta f = \int_t^{t+\Delta t} d\tilde{t} \int d\tilde{\mathbf{r}} \int d\tilde{\mathbf{p}} \exp\left\{-\int_{\tilde{t}}^{t+\Delta t} dt' \sigma^t v^{s'}\right\} \Phi(\mathbf{r},\mathbf{p},t+\Delta t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})$$
$$\times \left[ \int d\mathbf{p}' \sigma(\tilde{\mathbf{p}},\mathbf{p}') v' f(\tilde{t},\tilde{\mathbf{r}},\mathbf{p}') - \delta(t-\tilde{t}) \int_{t-\Delta t}^{t} dt' \int d\mathbf{p}' \sigma(\tilde{\mathbf{p}},\mathbf{p}') v' f(t',\tilde{\mathbf{r}},\mathbf{p}') \right]. \quad (9)$$

The action of the finitely supported function (9) on an element of the test function space has the form

$$(\Delta f,\varphi) = \int_t^{t+\Delta t} d\tilde{t} \int d\tilde{\mathbf{r}} \int d\tilde{\mathbf{p}} \exp\left\{-\int_{\tilde{t}}^{t+\Delta t} dt' \sigma^t v^{s'}\right\} \varphi(t+\Delta t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}) \sigma^t(\tilde{p}) \tilde{v} u(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})$$
$$- \int_{t-\Delta t}^{t} dt' \int d\tilde{\mathbf{r}} \int d\tilde{\mathbf{p}} \exp\left\{-\int_{t}^{t+\Delta t} dt' \sigma^t v^{s'}\right\} \varphi(t+\Delta t,t,\tilde{\mathbf{r}},\tilde{\mathbf{p}}) \sigma^t(\tilde{p}) \tilde{v} u(t',\tilde{\mathbf{r}},\tilde{\mathbf{p}}). \quad (10)$$

The arrival of a particle into a phase volume element due to collisions is written in (10) in the form

$$\int d\mathbf{p}' \sigma(\mathbf{p},\mathbf{p}') v' f(t,\mathbf{r},\mathbf{p}') = \sigma^t(p) v u(t,\mathbf{r},\mathbf{p}).$$

We have introduced the following notation:

$$u(t,\mathbf{r},\mathbf{p}) \equiv \int d\mathbf{p}' w(\mathbf{p},\mathbf{p}') f(t,\mathbf{r},\mathbf{p}'), \quad w(\mathbf{p},\mathbf{p}') \equiv \sigma(\mathbf{p},\mathbf{p}') v' / \sigma^t v,$$

$$\varphi\left(\mathbf{r}^s(t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}), \mathbf{p}^s(t+\Delta t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})\right) \equiv \varphi(t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}).$$

By Rolle's theorem, there are $t^+ \in [t,t+\Delta t]$ and $t^- \in [t-\Delta t,t]$ such that

$$(\Delta f,\varphi) = \int d\tilde{\mathbf{r}} \int d\tilde{\mathbf{p}} \sigma^t(\tilde{p}) \tilde{v} \Delta t \left[ u\left(t^+,\tilde{\mathbf{r}},\tilde{\mathbf{p}}\right) \exp\left\{-\int_{t^+}^{t+\Delta t} dt' \sigma^t v^{s'}\right\} \varphi\left(t+\Delta t,t^+,\tilde{\mathbf{r}},\tilde{\mathbf{p}}\right) \right.$$
$$\left. - u\left(t^-,\tilde{\mathbf{r}},\tilde{\mathbf{p}}\right) \exp\left\{-\int_{t}^{t+\Delta t} dt' \sigma^t v^{s'}\right\} \varphi(t+\Delta t,t,\tilde{\mathbf{r}},\tilde{\mathbf{p}}) \right]. \quad (11)$$

Therefore, expression (7) determines the solution of Eq. (1) with the first order of accuracy with respect to the parameter $\sigma^t v\Delta t$. In numerical calculations, the condition $\sigma^t v\Delta t \ll 1$ should be satisfied.

## 3. THE STATISTICAL INTERPRETATION

In most cases of practical importance, the external electron source $Q = Q(t,\mathbf{r},\mathbf{p})$ (see [19]) is constructed on the basis of processing and interpretation of dosimetry data. The primary physical meaning of such data reflects the parameters of ionization in detector's active zone and relative position of the radiation source. From these data, the number of particles produced per unit time in certain, possibly small but always finite volumes of the phase space, $\Delta\mathbf{r}_i\Delta\mathbf{p}_i$, $i = 1, \ldots, I$, is reconstructed. In addition, radiation measurements detect the number of pairs of charge carriers forming for a finite time $t_{n+1} - t_n = \Delta t > 0$, $n = 0, \ldots, N$, rather than the instant ionization rate. Measurements of this kind are not capable of distinguishing particles forming inside a volume $\Delta\mathbf{r}_i\Delta\mathbf{p}_i$ for a time $\Delta t$. Therefore, it is natural to specify the external source by a finite sum of elementary sources:

$$Q(t,\mathbf{r},\mathbf{p}) = \sum_n \delta(t - t_n)\sum_i q_{ni}\delta(\mathbf{r} - \mathbf{r}_i)\delta(\mathbf{p} - \mathbf{p}_i), \tag{12}$$

where $\mathbf{r}_i \in \Delta\mathbf{r}_i$, $\mathbf{p}_i \in \Delta\mathbf{p}_i$, and $q_{ni}$ is the number of particles forming inside the volume $\Delta\mathbf{r}_i\Delta\mathbf{p}_i$ for the time $t_{n+1} - t_n$.

In such definition, the function $Q(t,\mathbf{r},\mathbf{p})$ becomes a generalized function not only of coordinates and momenta but of time as well. The order of its singularity as a function of each variable is unity (see [24]). Therefore, the solution of the Cauchy problem for Eqs. (1) and (2) with the initial data in the hyperplane $t = 0$ and such right-hand side is an ordinary function of time. The function $Q(t,\mathbf{r},\mathbf{p})$ is a finitely supported generalized function (see [24]). Indeed, since all $\mathbf{r}_i$ and $\mathbf{p}_i$ are chosen from a finite set, they belong to some bounded subdomain $D$ of the phase space and all $t_n$ belong to a certain time interval. The action of the generalized function $Q(t,\mathbf{r},\mathbf{p})$ defined by formula (12) on any test function with a support located outside the domain $D$ or the time interval is zero.

If the source in Eq. (1) is specified in the form (12), then, for any $t = t_n$, the aggregate $f(t,\mathbf{r},\mathbf{p})\delta(t - t_n) + Q(t,\mathbf{r},\mathbf{p})$ in formula (6) can be represented in the form

$$f(t,\mathbf{r},\mathbf{p})\delta(t - t_n) + Q(t,\mathbf{r},\mathbf{p}) = \delta(t - t_n)\sum_k N_{n,k}\delta(\mathbf{r} - \mathbf{r}_k)\delta(\mathbf{p} - \mathbf{p}_k). \tag{13}$$

The right-hand side of (13) sums all electrons produced and not absorbed till the instant of time $t = t_n$, inclusively. The quantity $N_{n,k}$ expresses the number of electrons having the coordinates $\mathbf{r}_k, \mathbf{p}_k$ at the instant of time $t = t_n$. For electrons produced immediately at the instant of time $t = t_n$, we have the relationship $N_{n,k} = q_{nk}$. For electrons produced at an instant of time $t_1 < t_n$, this quantity is proportional to $\exp\left\{-\int_{t_1}^{t_n} dt'\sigma^t v^{s'}\right\}$.

Consider the solution of Eq. (1) and the equivalent integral equation (4) in the interval $[t_{n-1}, t_n]$ by the method of successive generations [26], assuming the solution $f(t,\mathbf{r},\mathbf{p})$ to be known at $t = t_{n-1}$. The solution of the kinetic equation in the zero generation in the interval $[t_{n-1}, t_n]$, $f_0 \equiv f_0(t,\mathbf{r},\mathbf{p})$, has the form

$$f_0 = \sum_k N_{n-1,k}\exp\left\{-\int_{n-1}^{t} dt''\sigma^t v\left(p^s(t'', t_{n-1}\mathbf{r}_k, \mathbf{p}_k)\right)\right\}\Phi(\mathbf{r},\mathbf{p},t,t_{n-1},\mathbf{r}_k,\mathbf{p}_k). \tag{14}$$

Following [16], each term of the sum (14) will be called a particle. The first-generation particle source is represented in the form

$$\hat{K}f_0 = \sum_k N_{n-1,k}\exp\left\{-\int_{n-1}^{t} dt''\sigma^t v\left(p_k^{s''}\right)\right\}\delta\left(\mathbf{r} - \mathbf{r}_k^s\right)\sigma v\left(\mathbf{p},\mathbf{p}_k^s\right), \tag{15}$$

where $p_{\mathbf{k}}^{s''} \equiv p^s(t'', t_{n-1}, \mathbf{r}_k, \mathbf{p}_k)$, $\mathbf{r}_k^s(t) \equiv \mathbf{r}^s(t, t_{n-1}, \mathbf{r}_k, \mathbf{p}_k)$, and $\mathbf{p}_k^s(t) \equiv \mathbf{p}^s(t, t_{n-1}, \mathbf{r}_k, \mathbf{p}_k)$.

Suppose that a random variable $a_0$ is uniformly distributed in the interval [0, 1]. Define the random variable $\tau_k^0$ as the root of the equation

$$\exp\left\{-\int_{t_{n-1}}^{t_{n-1}+\tau_k^0} dt''\sigma^t v(p_{0k}^{s''})\right\} = a_0. \tag{16}$$

In [18], the following formulas for the mathematical expectations have been derived:

$$M\left[\Theta\left(\exp\left\{-\int_{t_{n-1}}^{t} dt''\sigma^t v\left(p_k^{s''}\right)\right\} - a_0\right)\right] = \exp\left\{-\int_{t_{n-1}}^{t} dt''\sigma^t v\left(p_k^{s''}\right)\right\}, \tag{17}$$

$$M\left[\delta\left(\tau_k^0 - (t - t_{n-1})\right)\right] = \sigma^t v\left(p_k^s(t)\right)\exp\left\{-\int_{t_{n-1}}^{t} dt''\sigma^t v\left(p_k^{s''}\right)\right\}. \tag{18}$$

The mathematical expectation of the random variable $\tau_k^0$ is the lifetime of the particle with the index $k$.

Suppose that the random variable $\mathbf{p}_k^1$ is distributed with a probability density $w\left(\mathbf{p}, \mathbf{p}_k^s\right)$. It is obvious that

$$M\left[\delta\left(\mathbf{p} - \mathbf{p}_k^1\right)\right] = w\left(\mathbf{p}, \mathbf{p}_k^s\right). \tag{19}$$

The random variables $a_0$ and $\mathbf{p}_k^1$ are independent. We have the following possible statistical interpretation of the zero-generation particle distribution function (14) and the first-generation particle source (15):

$$f_0^{st} = \sum_k N_{n-1,k}\Theta\left(\exp\left\{-\int_{t_{n-1}}^{t} dt''\sigma^t v\left(p_k^{s''}\right)\right\} - a\right)\delta\left(\mathbf{r} - \mathbf{r}_k^s(t)\right)\delta\left(\mathbf{p} - \mathbf{p}_k^s(t)\right), \tag{20}$$

$$\hat{K}f_0^{st} = \sum_k N_{n-1,k}\delta\left(\tau_k^0 - (t - t_{n-1})\right)\delta\left(\mathbf{r} - \mathbf{r}_k^s\right)\delta\left(\mathbf{p} - \mathbf{p}_k^1\right), \tag{21}$$

where

$$M\left[f_0^{st}\right] = f_0, \quad M\left[\hat{K}f_0^{st}\right] = \hat{K}f_0. \tag{22}$$

The mathematical expectation in formulas (22) is calculated from the quantities $a_0$ and $\mathbf{p}_k^1$.

Consider the expression for distribution function (20) at $t = t_n$:

$$f_0^{st}(t_n, \mathbf{r}, \mathbf{p}) = \sum_k N_{nk}\Theta\left(\tau_k^0 - \Delta t\right)\delta\left(\mathbf{r} - \mathbf{r}_k^s(t_n)\right)\delta\left(\mathbf{p} - \mathbf{p}_k^s(t_n)\right).$$

It should be noted that, due to (16),

$$\Theta\left(\exp\left\{-\int_{t_{n-1}}^{t_n} dt''\sigma^t v\left(p_{0k}^{s''}\right)\right\} - a\right) = \Theta\left(\tau_k^0 - \Delta t\right).$$

Also consider

$$\int_{t_{n-1}}^{t_n} \hat{K}f_0^{st} dt = \sum_k N_{n-1,k}\Theta\left(\Delta t - \tau_k^0\right)\delta\left(\mathbf{r} - \mathbf{r}_k^s\left(t_{n-1} + \tau_k^0\right)\right)\delta\left(\mathbf{p} - \mathbf{p}_k^1\left(t_{n-1} + \tau_k^0\right)\right).$$

In the framework of the algorithm of the first-order accuracy with respect to time, which is used for solving the equation of particles' motion, the current values of the coordinates and momenta at the instants of time $t_{n-1}+\tau_k^0$ and $t_n$ coincide. Consider the expression

$$\delta(t-t_n)\Bigg\{\sum_i q_{ni}\delta(\mathbf{r}-\mathbf{r}_i)\delta(\mathbf{p}-\mathbf{p}_i) + \sum_k N_{n-1,k}\delta\left(\mathbf{r}-\mathbf{r}_k^s(t_n)\right)\left[\Theta\left(\tau_k^0-\Delta t\right)\delta\left(\mathbf{p}-\mathbf{p}_k^s(t_n)\right)+\Theta\left(\Delta t-\tau_k^0\right)\delta\left(\mathbf{p}-\mathbf{p}_k^1(t_n)\right)\right]\Bigg\}. \tag{23}$$

If the electron source is defined by formula (12), then aggregate (23) is a statistical analog of the last multiplier in the integrand on the right of formula (7):

$$f(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})\delta(t-\tilde{t})+Q(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})+\delta(t-\tilde{t})\int\limits_{t-\Delta t}^{t}dt'\int d\mathbf{p}'\sigma(\tilde{\mathbf{p}},\mathbf{p}')v'f(t',\tilde{\mathbf{r}},\mathbf{p}').$$

Consider the particle with a number $k$, having at the instant of time $t=t_{n-1}$ the coordinates $\mathbf{r}=\mathbf{r}_k$ and $\mathbf{p}=\mathbf{p}_k$ in the phase space. Until the instant of time $t=t_n$, it moves without collisions and takes the coordinates $\mathbf{r}_k^s(t_n)$ and $\mathbf{p}_k^s(t_n)$, which are calculated by solving the equations of motion (5). At the instant of time $t=t_n$, the value of the random variable $\tau_k^0$, the particle lifetime, is raffled. If $\tau_k^0$ exceeds the time step $\Delta t=t_n-t_{n-1}$, then the particle conserves the momentum $\mathbf{p}_k^s(t_n)$. If $\tau_k^0$ is smaller than $\Delta t$, then the particle is excluded from further consideration. In this case, the momentum after scattering is raffled and a new particle with the momentum $\mathbf{p}_k^1(t_n)$ is generated. Particles produced by the source at the instant of time $t=t_n$ are also added. Thus, in the time interval $(t_n,t_{n+1})$, there are particles emitted by the source at $t=t_n$ (the first term on the right of (23)), particles existing at $t=t_{n-1}$ and not scattered for the time $\Delta t=t_n-t_{n-1}$ (the second term), and particles produced due to scattering (the third term).

It should be noted that this model enables one to consider various types of particles, e.g., photons, electrons, and positrons in a cascade. The scattering of a particle of a given type can result in the formation of one particle of the same type (excitation of an atom by an electron, photon coherent scattering), two particles of the same type (impact ionization of an atom with emission of two identical indistinguishable electrons), one particle of a different type (photoabsorption with emission of a recoil electron, positron annihilation), and two particles of different types (electron or positron bremsstrahlung, Compton scattering, electron–positron pair formation).

Only one scattering of a particle in a time step, $\sigma^t v_k^s(t_n)\Delta t$, must be so small that the second scattering, whose probability is proportional to $\left(\sigma^t v_k^s(t_n)\Delta t\right)^2$, may be neglected. This follows both from the physics of scattering and from estimate (11).

## 4. MULTIPLE SCATTERING IN A TIME STEP

Consider formulas (14), (15), (20), and (21) in the interval $((t_n,t_{n+1}))$:

$$f_0=\sum_k N_{n,k}\exp\left\{-\int\limits_{t_n}^{t}dt''\sigma^t v\left(p^s(t'',t_n\mathbf{r}_k,\mathbf{p}_k)\right)\right\}\Phi(\mathbf{r},\mathbf{p},t,t_n,\mathbf{r}_k,\mathbf{p}_k),$$

$$\hat{K}f_0=\sum_k N_{n,k}\exp\left\{-\int\limits_{t_n}^{t}dt''\sigma^t v\left(p^s(t'',t_n\mathbf{r}_k,\mathbf{p}_k)\right)\right\}\delta\left(\mathbf{r}-\mathbf{r}_k^s(t)\right)\sigma v\left(\mathbf{p},\mathbf{p}_k^s(t)\right),$$

$$f_0^{st} = \sum_k N_{n,k}\Theta\left(\exp\left\{-\int_{t_n}^{t} dt''\sigma^t v\left(p_k^{s''}\right)\right\} - a_0\right)\delta\left(\mathbf{r}-\mathbf{r}_k^s(t)\right)\delta\left(\mathbf{p}-\mathbf{p}_k^s(t)\right),$$

$$\hat{K}f_0^{st} = \sum_k N_{n,k}\delta\left(\tau_k^0 - (t-t_n)\right)\delta\left(\mathbf{r}-\mathbf{r}_k^s\right)\delta\left(\mathbf{p}-\mathbf{p}_k^1\right),$$

denoting $\mathbf{r}_k^s(t) \equiv \mathbf{r}^s(t,t_n,\mathbf{r}_k,\mathbf{p}_k)$ and $\mathbf{p}_k^s(t) \equiv \mathbf{p}^s(t,t_n,\mathbf{r}_k,\mathbf{p}_k)$.

Calculating the first-generation electron distribution function $f_1$ yields

$$f_1 = \sum_k N_{n,k}\int_{t_n}^{t} d\tilde{t}\int d\tilde{\mathbf{r}}\int d\tilde{\mathbf{p}}\sigma v_k^s(\tilde{t})\, w\left(\tilde{\mathbf{p}},\mathbf{p}_k^s(\tilde{t})\right)\Phi\left(\mathbf{r},\mathbf{p},t,\tilde{t},\mathbf{r}_k^s(\tilde{t}),\tilde{\mathbf{p}}\right)$$
$$\times\exp\left\{-\int_{t_n}^{\tilde{t}} dt''\sigma^t v\left(p_k^s(t'')\right) - \int_{\tilde{t}}^{t} dt'\sigma^t v^s\left(t',\tilde{t},\mathbf{r}_k^s(\tilde{t}),\tilde{\mathbf{p}}\right)\right\}.$$

Due to (17)–(19),

$$f_1 = \sum_k N_{n,k}\int_{t_n}^{t} d\tilde{t}\int d\tilde{\mathbf{p}}\int_0^1 da_0\int d\mathbf{p}_k^1\exp\left\{-\int_{\bar{t}}^{t} dt''\sigma^t v\left(p^s\left(t',\tau,\mathbf{r}_k^s(\tilde{t}),\tilde{\mathbf{p}}\right)\right)\right\}$$
$$\times\delta\left(\tau_k^0-(\tilde{t}-t_n)\right)\delta\left(\tilde{\mathbf{p}}-\mathbf{p}_k^1\right)\Phi\left(\mathbf{r},\mathbf{p},t,\tilde{t},\mathbf{r}_k^s(\tilde{t}),\tilde{\mathbf{p}}\right).$$

Calculating the integrals with respect to the inner variables, we obtain

$$f_1 = \sum_k N_{n,k}\int_0^1 da_0\int d\mathbf{p}_k^1\Theta\left(t-t_n-\tau_k^0\right)\exp\left\{-\int_{t_n+\tau_k^0}^{t} dt''\sigma^t v\left(p^s\left(t',t_n+\tau_k^0,\mathbf{r}_k^s\left(t_n+\tau_k^0\right),\mathbf{p}_k^1\right)\right)\right\}$$
$$\times\Phi\left(\mathbf{r},\mathbf{p},t,t_n+\tau_k^0,\mathbf{r}_k^s\left(t_n+\tau_k^0\right),\mathbf{p}_k^1\right).$$

Hence,

$$f_1 = M\left[\sum_k N_{n,k}\Theta\left(t-t_n-\tau_k^0\right)\exp\left\{-\int_{t_n+\tau_k^0}^{t} dt''\sigma^t v\left(p^s\left(t',t_n+\tau_k^0,\mathbf{r}_k^s\left(t_n+\tau_k^0\right),\mathbf{p}_k^1\right)\right)\right\}\right.$$
$$\left.\times\,\Phi\left(\mathbf{r},\mathbf{p},t,t_n+\tau_k^0,\mathbf{r}_k^s\left(t_n+\tau_k^0\right),\mathbf{p}_k^1\right)\right]. \qquad (24)$$

Suppose that the random variable $a_1$ is uniformly distributed in the interval [0, 1] and is independent of $a_0$ and the random variable $\tau_k^1$ is a root of the equation

$$\exp\left\{-\int_{t_n+\tau_k^0}^{t_n+\tau_k^0+\tau_k^1} dt''\sigma^t v\left(p^s\left(t'',t_n+\tau_k^0,\mathbf{r}_k^s\left(t_n+\tau_k^0\right),\mathbf{p}_\mathbf{k}^1\right)\right)\right\} = a_1.$$

Then we have the following possible statistical interpretation of the first-generation particle distribution function:

$$f_1^{st} = \sum_k N_{n,k}\Theta\left(t-t_n-\tau_k^0\right)\Theta\left(t_n+\tau_k^0+\tau_k^1-t\right)\Phi\left(\mathbf{r},\mathbf{p},t,t_n+\tau_k^0,\mathbf{r}_k^s\left(t_n+\tau_k^0\right),\mathbf{p}_k^1\right),$$

where $M\left[f_1^{st}\right] = f_1$ .

The "statistical" first-generation particle distribution function (24) can be obtained by the direct integration of their "statistical" source:

$$f_1^{st}(t) = \int_{t_n}^{t} d\tilde{t} \int d\tilde{\mathbf{r}} \int d\tilde{\mathbf{p}} \hat{K} f_0^{st}(\tilde{t}, \tilde{\mathbf{r}}, \tilde{\mathbf{p}}) \exp\left\{ -\int_{\tilde{t}}^{t} dt' \sigma^{t} v^{s'} \right\} \delta\left(\mathbf{r} - \mathbf{r}^{s}\right) \delta\left(\mathbf{p} - \mathbf{p}^{s}\right).$$

Hence, the statistical distribution functions of particles of further generations, $f_m^{st}$, can be constructed without consideration of deterministic functions. Moreover, the relationship

$$M\left[f_m^{st}\right] = f_m,$$

holds, where $f_m$ is the deterministic distribution function of the $m$th generation of particles. In each generation, two new random variables $a_m$ and $\mathbf{p}_k^m$, which are mutually independent and independent of their counterparts in previous generations, arise.

The infinite sum of the distribution functions of particles in all generations,

$$f = \sum_{m=0}^{\infty} f_m$$

yields a sufficiently precise solution of Eq. (1) for any $t > t_n$.

Therefore,

$$f = M\left[f^{st}\right] = \sum_{m=0}^{\infty} M\left[f_m^{st}\right]. \tag{25}$$

Consider $f^{st}(t_{n+1}, \mathbf{r}, \mathbf{p})$. It is theoretically possible that series (25), which defines it, contains an infinite number of terms. In calculations, it means that, at least for one combination of indices $n$ and $k$, the condition $\sum_{m=0}^{\infty} \tau_k^m < \Delta t$ is satisfied. The probability of such an event is negligibly small. Therefore, $f^{st}(t_{n+1}, \mathbf{r}, \mathbf{p})$ is expressed by a finite sum and is a finitely supported generalized function of the variables $\mathbf{r}$ and $\mathbf{p}$.

Thus, the electron distribution function $f \equiv f(t, \mathbf{r}, \mathbf{p})$ at an arbitrary time point is a finitely supported generalized function depending on the parameter $t$ and a finite number of random parameters $a_m$ and $\mathbf{p}_k^m$.

## 5. SPECIFICITIES OF SIMULATION OF IONIZED MEDIA OF LARGE VOLUME

Some practical problems require the consideration of the propagation of a pulsed photon flux and the cascade produced by it in a gaseous medium to long distances [27]. In this case, it is advisable to pass from the variables $(t, \mathbf{r}, \mathbf{p})$ to the variables $(\xi, \mathbf{r}, \mathbf{p})$, where $\xi = t - r/c$ is the proper time of the photon radiation leading edge.

In the proper time, the Cauchy problem (1) turns into the Goursat problem [25] for the equations [27]

$$\begin{aligned} (1 - \langle \boldsymbol{\beta}, \mathbf{e}_r \rangle) \frac{\partial f}{\partial \xi} + \operatorname{div}_{\mathbf{r}}(\mathbf{v} f) - e \operatorname{div}_{\mathbf{p}}\left[ (\mathbf{E} + [\boldsymbol{\beta}, \mathbf{H}]) f \right] + \sigma^{t} v f \\ = Q(\xi, \mathbf{r}, \mathbf{p}) + \int d\mathbf{p}' \sigma(\mathbf{p}, \mathbf{p}') v' f(\mathbf{p}'), \end{aligned} \tag{26}$$

$$\operatorname{curl}\mathbf{H} = \frac{1}{c}\left[\mathbf{e}_r, \frac{\partial \mathbf{H}}{\partial \xi}\right] + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial \xi} + \frac{4\pi}{c}\mathbf{j}, \quad \operatorname{curl}\mathbf{E} = \frac{1}{c}\left[\mathbf{e}_r, \frac{\partial \mathbf{E}}{\partial \xi}\right] - \frac{1}{c}\frac{\partial \mathbf{H}}{\partial \xi},$$

where $\mathbf{e}_r$ is a unit vector in the radial direction.

The kinetic equation of system (26) is equivalent to the following integral equation (see [27]):

$$f = \int_0^{\xi} d\tilde{t} \int d\tilde{\mathbf{r}} \int d\tilde{\mathbf{p}} \exp\left\{-\int_{t_1}^{\tau} dt' \sigma^t v^{s'}\right\}\left[Q(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}) + \int d\mathbf{p}' \sigma(\tilde{\mathbf{p}},\mathbf{p}') v' f(\tilde{t},\tilde{\mathbf{r}},\mathbf{p}')\right] \times \int ds \delta\left(\xi - \xi^s(s,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})\right) \Phi(\mathbf{r},\mathbf{p},s,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}), \tag{27}$$

where the functions $\tau^s \equiv \tau^s(s,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})$, $\mathbf{r}^s \equiv \mathbf{r}^s(s,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})$, $\mathbf{p}^s \equiv \mathbf{p}^s(s,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})$, $\mathbf{r}_1^{s'} \equiv \mathbf{r}^s(s',\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})$, and $\mathbf{p}_1^{s'} \equiv \mathbf{p}^s(s',\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}})$ are the solutions to the equations of motion

$$d\xi^s/ds = 1 - \langle \boldsymbol{\beta}^s, \mathbf{e}_r \rangle, \quad d\mathbf{r}^s/ds = \mathbf{v}^s \, d\mathbf{p}^s/ds = e\left(\mathbf{E}^s + \left[\boldsymbol{\beta}^s, \mathbf{H}^s\right]\right) \tag{28}$$

with the initial data $\xi^s\big|_{s=\tilde{t}} = \tilde{t}$, $\mathbf{r}^s\big|_{s=\tilde{t}} = \tilde{\mathbf{r}}$, and $\mathbf{p}^s\big|_{s=\tilde{t}} = \tilde{\mathbf{p}}$.

The variable $s$ in (28) is used for the parameterization of the particle's path in the phase space $(\mathbf{r},\mathbf{p}) = \mathbb{R}_r^3 \times \mathbb{R}_p^3$. The physical meaning of this variable is the laboratory time, i.e., $s = t$, which is suggested by the solution of the first equation (28):

$$\xi^s = s - r^s/c - (\tilde{t} - \tilde{r}/c), \tag{29}$$

and the form of the remaining equations of motion.

Taking into account the physical meaning of the path parameter and solution (29), we transform Eq. (27) to

$$f = \int_0^{\xi} d\tilde{t} \int d\tilde{\mathbf{r}} \int d\tilde{\mathbf{p}} \exp\left\{-\int_{\tilde{t}}^{\xi} dt' \sigma^t v^{s'}\right\}\left[Q(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}) + \int d\mathbf{p}' \sigma(\tilde{\mathbf{p}},\mathbf{p}') v' f(\tilde{t},\mathbf{r}_1,\mathbf{p}')\right] \times \int dt \delta\left(\xi - \left(t - r^s/c\right) + (t_1 - r_1/c)\right) \Phi(\mathbf{r},\mathbf{p},t,\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}). \tag{30}$$

Equations (30) in the proper time and Eqs. (4) in the laboratory time coincide up to the change of variables $t,\mathbf{r} \to \xi,\mathbf{r}$. Therefore, the motion and scattering of particles may be considered in the laboratory time, if one uses an algorithm identical to that described in the previous section. The only addition is that the particle lifetime in the laboratory time defined as a function of a random variable should be put into correspondence with an interval of the proper time with the aid of solution (29) of the first equation in (28).

## 6. SPECIFICITIES OF SIMULATION OF A CASCADE IN DENSE MATERIALS

Consider Eq. (3) in materials with such a density that the particle collision frequency substantially exceeds the plasma frequency while the electron lifetime is small as compared to the source duration. Then the influence of the self-consistent field on the particle motion may be neglected. Equation (4) takes the form

$$f = \int_0^{t} d\tilde{t} \int d\tilde{\mathbf{r}}\left[Q(\tilde{t},\tilde{\mathbf{r}},\tilde{\mathbf{p}}) + \int d\mathbf{p}' \sigma(\tilde{\mathbf{r}},\tilde{\mathbf{p}},\mathbf{p}') v' f(t,\tilde{\mathbf{r}},\mathbf{p}')\right] \times \exp\left\{-\int_{\tilde{t}}^{t} v dt' \sigma^t\left(\mathbf{r}^{s'},p\right)\right\} \delta(\mathbf{r} - \tilde{\mathbf{r}} - \mathbf{v}(\mathbf{p})(t - \tilde{t})). \tag{31}$$

In Eq. (31), it is taken into account that macroscopic scattering cross sections may depend on spatial coordinates. By the change of variables $\mathbf{v}(\mathbf{p})(t-\tilde{t}) = v(p)\boldsymbol{\Omega}(t-\tilde{t}) = \boldsymbol{\Omega}\varsigma$, $v(p)(t-\tilde{t}) = \zeta$, $t-\varsigma/v = \tilde{t}$, $v(p)(t'-\tilde{t}) = \varsigma'$, and $\varsigma - \varsigma' = \varsigma''$, Eq. (31) is reduced to the equation

$$f = \frac{1}{v}\int_0^{vt} d\varsigma \exp\left\{-\int_0^{\varsigma} d\varsigma'' \sigma^t(\mathbf{r}-\boldsymbol{\Omega}\varsigma'', p)\right\} \times \left[Q\left(t-\frac{\varsigma}{v}, \mathbf{r}-\boldsymbol{\Omega}\varsigma, \mathbf{p}\right) + \int d\mathbf{p}' \sigma(\mathbf{r}-\boldsymbol{\Omega}\varsigma, \mathbf{p}, \mathbf{p}') v' f\left(t-\frac{\varsigma}{v}, \mathbf{r}-\boldsymbol{\Omega}\varsigma, \mathbf{p}'\right)\right], \tag{32}$$

which coincides with the stationary integral equation of radiative transfer from the classical book [26] up to the limit of integration over the variable $\varsigma$. The integration with respect to $\varsigma$ from 0 to $vt$ means that, at the point $\mathbf{r}$, to the instant of time $t$, particles capable of passing the distance $vt$ gather.

Consider the solution to Eq. (32) by the successive-generation method. For the zero-generation particle distribution function, we have

$$f_0 = \frac{1}{v}\int_0^{vt} d\varsigma \exp\left\{-\int_0^{\varsigma} d\varsigma'' \sigma^t(\mathbf{r}-\boldsymbol{\Omega}\varsigma'', p)\right\} Q\left(t-\frac{\varsigma}{v}, \mathbf{r}-\boldsymbol{\Omega}\varsigma, \mathbf{p}\right). \tag{33}$$

In the overwhelming majority of practical problems, the lifetimes of particles in dense materials are small as compared to the radiation source duration and, therefore, to the characteristic time of variation in the unscattered radiation distribution functions. With respect to time and, therefore, the variable $\varsigma$, the source varies substantially more slowly than the exponential $\exp\left\{-\int_0^{\varsigma} d\varsigma'' \sigma^t(\mathbf{r}-\boldsymbol{\Omega}\varsigma'', p)\right\}$. Represent the source in the form $Q(t, \mathbf{r}, \mathbf{p}) = N(t)\bar{Q}(\mathbf{r}, \mathbf{p})$, separating the dependence of the integrand (33) on $t-\varsigma/v$ in the explicit form. Therefore, we have the following approximation:

$$f_0 \approx \frac{1}{v} N(t) \int_0^{vt} d\varsigma \exp\left\{-\int_0^{\varsigma} d\varsigma'' \sigma^t(\mathbf{r}-\boldsymbol{\Omega}\varsigma'', p)\right\} \bar{Q}(\mathbf{r}-\boldsymbol{\Omega}\varsigma, \mathbf{p}). \tag{34}$$

Consider the following approximation. Represent distribution function (34) in the form

$$f_0 = \frac{1}{v} N(t) \left[\int_0^{\infty} d\varsigma \exp\left\{-\int_0^{\varsigma} d\varsigma'' \sigma^t(\mathbf{r}-\boldsymbol{\Omega}\varsigma'', p)\right\} \bar{Q}(\mathbf{r}-\boldsymbol{\Omega}\varsigma, \mathbf{p}) - \int_{vt}^{\infty} d\varsigma \exp\left\{-\int_0^{\varsigma} d\varsigma'' \sigma^t(\mathbf{r}-\boldsymbol{\Omega}\varsigma'', p)\right\} \bar{Q}(\mathbf{r}-\boldsymbol{\Omega}\varsigma, \mathbf{p})\right]. \tag{35}$$

The distribution function in dense materials is calculated at times substantially exceeding the particle lifetimes, i.e., we have the inequality $\sigma^t vt \gg 1$. This means that the first term in (35) substantially exceeds the second term due to the rapid decrease in the exponential. In this case, the approximation takes the following form:

$$f_0 \approx \frac{1}{v} N(t) \int_0^{\infty} d\varsigma \exp\left\{-\int_0^{\varsigma} d\varsigma'' \sigma^t(\mathbf{r}-\boldsymbol{\Omega}\varsigma'', p)\right\} \bar{Q}(\mathbf{r}-\boldsymbol{\Omega}\varsigma, \mathbf{p}).$$

This approximation is violated near $t = 0$ at the times on the order of a few lifetimes of scattered particles. It should be noted that, in the exact formulation, the function $N(t)$ at $t = 0$ is zero.

The approximations introduced above hold in any subsequent generation; moreover, their accuracy increases with generations. This follows from the fact that the particle energy and, therefore, the path length on scattering do not increase—on the contrary, on the average, they decrease. Therefore, Eq. (32) is transformed to

$$\bar{f} = \int_0^\infty d\varsigma \exp\left\{-\int_0^\varsigma d\varsigma'' \sigma^t(\mathbf{r}-\boldsymbol{\Omega}\varsigma'', p)\right\} \times \left[\bar{Q}(\mathbf{r}-\boldsymbol{\Omega}\varsigma, \mathbf{p}) + \int d\mathbf{p}' \sigma(\mathbf{r}-\boldsymbol{\Omega}\varsigma, \mathbf{p}, \mathbf{p}') v' \bar{f}(\mathbf{r}-\boldsymbol{\Omega}\varsigma, \mathbf{p}')\right], \quad (36)$$

where $vf(t, \mathbf{r}, \mathbf{p}) \equiv N(t)\bar{f}(\mathbf{r}, \mathbf{p})$.

The algorithm for solving Eq. (36) is constructed by analogy with the algorithm for the nonstationary case. The simplification is in the following. In the stationary consideration, the particle lifetime is not compared with the time step. The fact of scattering is recorded, and the time is used only for the parameterization of the particle motion in the coordinate space.

## CONCLUSIONS

The condition for applicability of the single-scattering approximation in a time step is an obstacle for solving most problems of practical importance. Computations performed in this approximation have demonstrated the following [18]. For electrons with the energy as low as of a few electronvolts, the main collision process in a gas medium is elastic scattering. Its cross section exceeds by two orders of magnitude the cross sections of all inelastic processes. As a result, the study of, e.g., breakdown of gas in an external electric field requires the averaging of the elastic scattering process. Otherwise, the calculation is reduced to simulation of Brownian motion, on the background of which the ionization process is difficult to distinguish.

Difficulties also arise in the consideration of a cascade of high-energy particles with substantially different lifetimes. The time step in such a problem is determined by the cross section of ionizing electron collisions; each of them is accompanied by an averagely insignificant variation in the momentum. The calculation is reduced to simulation of electron moderation, and photon collisions become rare events.

The consideration of multiple collisions in a time step removes these difficulties. Despite that the amount of computation in a time step may dramatically increase because multiple processing of collisions of a certain fraction of particles is necessary, the consideration of the main process is not lost on the background of less significant processes. The time step may be chosen so that the main process be explicitly separated.

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